Super-resolution Radar

Reinhard Heckel

IBM Research (before: ETH Zurich)

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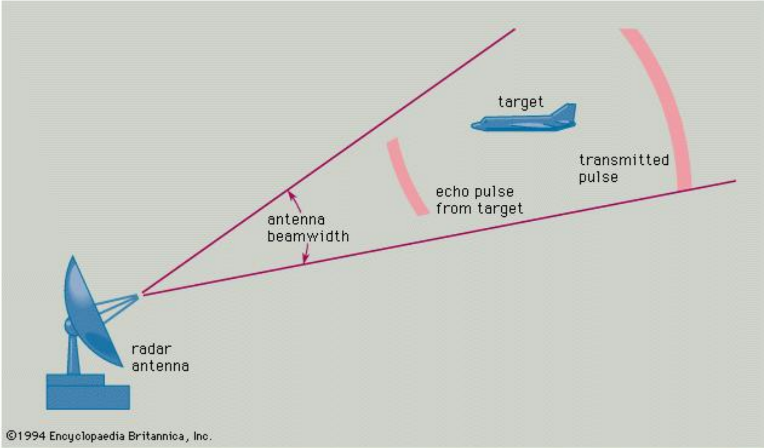
Joint work with:
V. Morgenshtern (Stanford), and M. Soltanolkotabi (Berkeley)

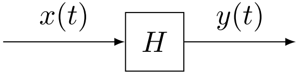
Motivation: Radar



Goal: determine location and velocities of objects

Model

Time-varying linear system H:



$$y(t) = \iint s_H(\tau, \nu) x(t - \tau) e^{i2\pi\nu t} d\nu d\tau$$

Goal: Determine $s_H(\tau, \nu)$ from response $y(t)$ to known probing signal $x(t)$

Model

- Spreading function consists of S point scatterers, that correspond to moving targets:

$$s_H(\tau, \nu) = \sum_{j=1}^{S} b_j \delta(\tau - \tau_j)\delta(\nu - \nu_j)$$

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- Determine the triplets (b_j, τ_j, ν_j) from I/O-measurement

Band and time-limitation

In practice: $x(t)$ bandlimited to $[0, B)$ and $y(t)$ timelimited to $[0, T)$

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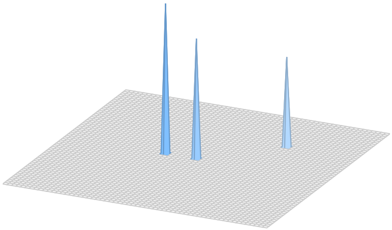
$s_H(\tau, \nu)$



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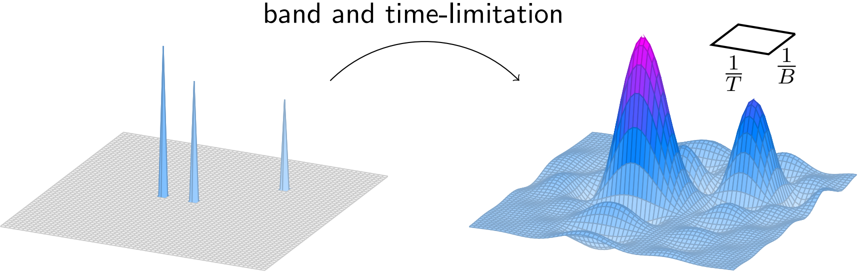

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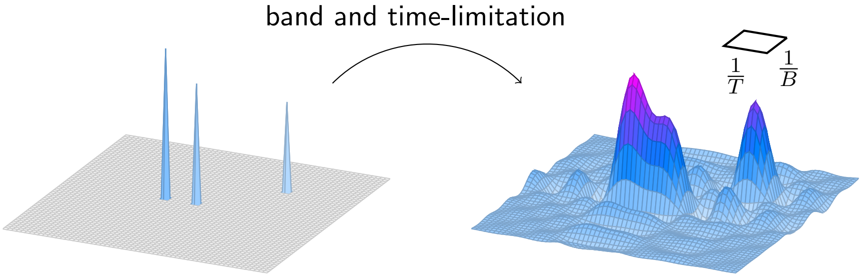

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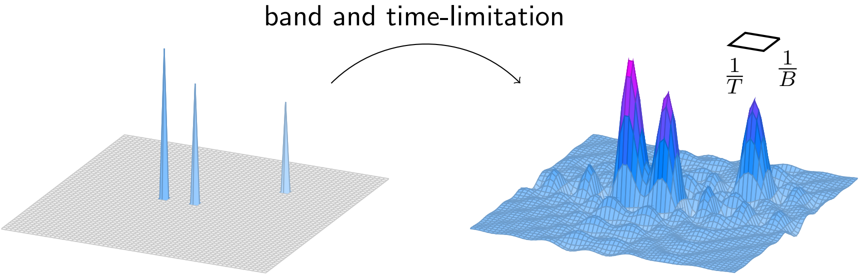

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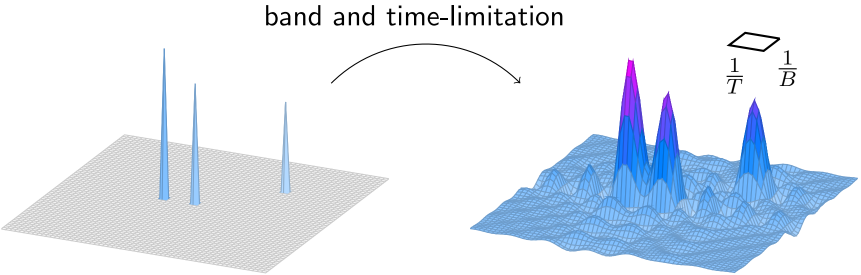

Resolution achieved by classic Radar via matched filtering is $\left(\frac{1}{B}, \frac{1}{T}\right)$!

Compressed sensing Radar [*Herman & Stromer, 2009*]

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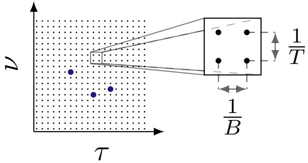
Suppose the (τ_j, ν_j) lie on a $\left(\frac{1}{B}, \frac{1}{T}\right)$-grid



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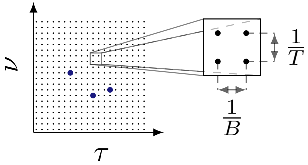
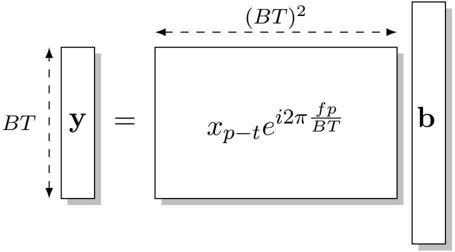


Recovery is a sparse signal recovery problem:



Recovery via ℓ_1 minimization provably succeeds provided that $S \leq c(BT)/\log^4(BT)$ [*Krahmer et al. 2014*]

The super-resolution Radar problem

Sampling leads to the following I/O relation:

$$y_p = \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}} \mathrm{IDFT}(\mathrm{DFT}(\{x_{\ell-p}\}) e^{i2\pi k \frac{\tau_j}{T}}), \quad p = 0, ..., BT-1$$

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Super resolution Radar problem: Determine the continuous time-frequency shifts τ_j, ν_j from the samples y_p

This talk: Provably recovery of the (τ_j, ν_j) via convex optimization

Special case: Frequency shifts only

If $\tau_j = 0$, the problem reduces to a line spectral estimation problem:

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Recovery approaches:

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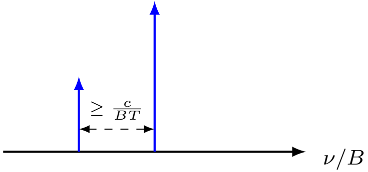
Recovery approaches:

- **Classical:** Proney's method
- **Modern:** Recovery via convex optimization [*Candès, Fernandez-Granda, 2014*]: Recovery is possible provided the minimum separation condition holds:

$$\left|\frac{\nu_j}{B} - \frac{\nu_i}{B}\right| \geq \frac{4}{BT}, \quad \text{for all } j \neq i$$



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- **Compressed sensing:** RIP guarantees that the energy of *all* sparse signals is preserved
- **Super-resolution:** Suppose the ν_j are at equidistant positions, and S is large

$$y_p = e^{i2\pi p\frac{\nu_j}{B}} \; b_j$$

Energy is only preserved if the distance between the ν_j is sufficiently large!

Recovery results for the super-resolution Radar problem

Main result

- Random probing signal: x_ℓ i.i.d. $\mathcal{N}(0,1)$
- Random signs: sign of b_n is i.i.d. uniform on the complex unit sphere

Theorem

The $(\tau_j, \nu_j, b_j), j = 1, ..., S$, can be recovered by solving a semidefinite program with probability $\geq 1 - \delta$ if

$$|\tau_j - \tau_i| \geq \frac{5}{B} \text{ or } |\nu_j - \nu_i| \geq \frac{5}{T}, \text{ for all } i \neq j$$

and if

$$S \leq cBT \log^{-3}\left(\frac{(BT)^6}{\delta}\right)$$

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- Essentially optimal

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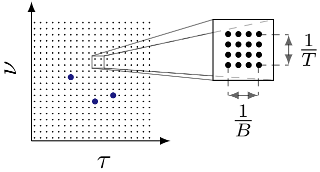
- Super-resolution Radar on a grid
  - Suppose the (τ_j, ν_j) lie on a **fine grid:**



  - Our results guarantee success of ℓ_1-minimization

Future work

Implementation in hardware:

Conclusion

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