A Compressive Sensing Perspective on Random Sensor Array

Based on Lawrence Carin’s papers:

Presented by Afsaneh Asaei
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Motivation

- Both random arrays and CS are manifested by the goal of realizing high-resolution data via a relatively small number of measurements.

- Goal:
  - Elucidate the connections between the two fields
  - Exploit the existing CS theory to quantify and improve the performance of random arrays
Compressive sensing perspective on

1. Sensor array projections
2. Aperture size and angular resolution
3. Side-lobe level
4. Signal estimation algorithms and performance bounds
5. Future directions in array processing
In a nutshell

- **CS** is sensing via dimensionality reduction
  - Dimensionality reduction naturally happens in many problems. So, we can leverage the CS theory and algorithms.
CS background theory

- **Measurement model** 
  \[ y = \Phi x + z \]

- **RIP**
  \[ (1 - \delta_s) \|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_s) \|x\|_2^2 \]

- **Reconstruction**
  \[ \min \|x\|_1 \quad \text{subject to} \quad \|y - \Phi x\|_2 \leq \varepsilon \]

- **Performance**
  \[ \|x^* - x\|_2 \leq C_0 s^{-1/2} \|x - x_s\|_1 + C_1 \varepsilon \]
  \[ m \geq C_3 s \mu^2 \left( \log n \right)^4 \]
  \[ s < \frac{1}{2} \left( \mu_c^{-1} + 1 \right) \]
  \[ \mu = \sqrt{n} \max_{i,j} \left| \Phi_{i,j} \right| \]
  \[ \mu_c = \max_{i \neq j} \left| \Phi_i^* \Phi_j \right| \]
Sensor array scenario

- General sensor array construction with a general pattern distributed in an arbitrary linear, isotropic environment

- For simplicity, here
  - Electromagnetic waves
  - Antennas are point (isotropic) radiators and receivers

- Goal: DOA estimation of point sources
Compressive sensing perspective on

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Array measurements as near-orthogonal projections

Far-zone source current

\[ q(r = R, \theta = \pi / 2, \varphi), \quad \varphi \in [0, 2\pi] \]

Electric field as observed at the \( i^{th} \) receiver at

\[ e(r_i) = \int_0^{2\pi} d\phi g(r_i; \phi)q(\phi) \]

Goal: Examine the relationship between the projection associated to two receivers at \( i \) and \( j \), we consider the reciprocity theorem

\[ q_1 = \delta(r - r_i) \]
\[ q_2 = q_{PC} = g^*(r_j; \varphi) \]

\[ e_{PC}(r_{obs} = r_i; r_{source} = r_j) = \int_0^{2\pi} d\phi g(r_i; \varphi)g^*(r_j; \varphi) \approx 0^\circ \quad \text{for} \quad \|r_i - r_j\|_2 \geq \lambda / 2 \]

\( ^0 \) insights from phase conjugation or time reversal
Minimum mutual coherence

A few radiators are distributed over \( \varphi \in [0, 2\pi] \)

\[
q(\varphi) = \sum_{k=1}^{s} w_k \delta(\varphi - \varphi_k^*) + v(\varphi)
\]

The projections manifested by the green’s function in free-space

\[
g(r; \varphi) = \exp(-j2\pi R / \lambda) \exp(-j2\pi r \cos(\varphi) / \lambda) / R
\]

Hence, the projection has constant amplitude with respect to \( \varphi \)
and therefore is maximally spread out
Compressive sensing perspective on

1. Sensor array projections
   - Natural projections manifested by the media Green’s function are consistent with the orthogonal, minimum mutual coherence projections associated with CS
Compressive sensing perspective on

1. Sensor array projections
2. Aperture size and angular resolution
3. Side-lobe level
4. Signal estimation algorithm
5. Performance bounds
6. Future directions in array processing
CS insights on high resolution projections

$q$ is only nonzero over a contiguous extent of angles

\[ q(r = R, \theta = \pi/2, \varphi), \quad \varphi \in S_{\varphi} \]

we again consider reciprocity which yields

\[ e_{PC}(r_{\text{obs}} = r_i; r_{\text{source}} = r_j) = \int_{\varphi \in S_{\varphi}} d\varphi g(r_i; \varphi)g^*(r_j; \varphi) \]

from aperture theory, for small $|S_{\varphi}|$ and for array in vacuum

\[ e_{PC}(r_{\text{obs}} = r_i; r_{\text{source}} = r_j) \approx 0^\circ \quad \text{for} \quad \|r_i - r_j\|_2 \geq \lambda / |S_{\varphi}| \]
Compressive sensing perspective on

2. Aperture size and angular resolution
   - The size of the aperture must be very large to achieve accurate CS projections over narrow resolution
Compressive sensing perspective on

1. Sensor array projections
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5. Future directions in array processing
Consider a linear sensor array with uniform inter-element spacing $\Delta_x$ reside in vacuum, the $i^{th}$ column of $\Phi$ is a sampled Fourier basis function, at angular frequency $\omega_i$

$$\begin{bmatrix} \Phi_{1,i}, \Phi_{2,i}, \ldots, \Phi_{m,i} \end{bmatrix}^T = [1, \omega_i, \omega_i^2, \ldots, \omega_i^{m-1}]^T,$$

$$\omega_i = \exp[-j2\pi \frac{\Delta_x}{\lambda} \cos(\varphi_i)]$$

$\Delta_x = \frac{\lambda}{2}$ achieves orthogonal projections on the source currents while removing the ambiguities in the angular frequencies: the later manifest “grating lobes” in the array response
Coherence

Consider a general array capture a far-zone source, the $i^{th}$ column of $\Phi$

$$\Phi_i = [g(r_1, \varphi_i), g(r_2, \varphi_i), ..., g(r_m, \varphi_i)]^T$$

$\langle \Phi_i, \Phi_j \rangle$ corresponds to the degree an array pointed at angle $\varphi_i$ “leaks” energy into the angular bin $\varphi_j$. Hence the coherence

$$\mu = \max_{i \neq j} |\langle \Phi_i, \Phi_j \rangle|$$

measures the maximum sidelobe levels for the array systems

From the RIP, when interested in accurately observing $s$ sources, the projection matrix should be designed such that any $2s$ columns should be nearly orthonormal which indicates also an explicit link with the systems resolution. To reduce the coherence, the array should be constructed to manifest random projections.
Compressive sensing perspective on

3. Side-lobe level
   - CS provides an explicit link between the array performance and the maximum side-lobe level through the coherence parameter
Compressive sensing perspective on

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Bearing estimation set-up

- Linear array in free space operating at 3GHz
- 4 m array aperture sampled uniformly with $\lambda/2$ spacing with 81 antennas

\[
\Phi_i = \frac{1}{m}[1, \exp(j \pi \cos \varphi_i),..., \exp(j(m-1)\pi \cos \varphi_i)]^T
\]

\[
\Phi_{i+1} = \frac{1}{m}[1, \exp(j \pi \cos(\varphi_i + \Delta_i),..., \exp(j(m-1)\pi \cos(\varphi_i + \Delta_i))]^T
\]

\[
\langle \Phi_i, \Phi_{i+1} \rangle = \frac{1}{m^2} \sum_{l=1}^{m} \exp\{j(l-1)\pi[\cos(\varphi_i + \Delta_i) - \cos(\varphi_i)]\} 
\]
Angular resolution

Fig. 3. Angle-bin decomposition based on an 81-element uniformly sampled array, with $\lambda/2$ inter-element spacing at 3 GHz (4 meter aperture). (a) the symbols denote the center of each angular bin; (b) angular width $\Delta_\theta$ of each bin, in degrees.
Coherence

TABLE I

COHERENCE FOR TWO ARRAY DESIGNS, AS A FUNCTION OF THE NUMBER OF ANTENNAS $m$. IN ONE DESIGN THE ANTENNAS ARE PLACED UNIFORMLY, WITH $\lambda/2$ INTER-ELEMENT SPACING, AND IN THE SECOND CASE THE ANTENNAS ARE PLACED RANDOMLY OVER A 4 METER APERTURE.

<table>
<thead>
<tr>
<th></th>
<th>$m = 21$</th>
<th>$m = 31$</th>
<th>$m = 41$</th>
<th>$m = 51$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda/2$ Spacing</td>
<td>0.915</td>
<td>0.820</td>
<td>0.698</td>
<td>0.558</td>
</tr>
<tr>
<td>Random Spacing</td>
<td>0.326</td>
<td>0.296</td>
<td>0.264</td>
<td>0.248</td>
</tr>
</tbody>
</table>
Based on 100,000 runs, the probability that Lasso [24] successfully inferred the correct underlying sparse signal $x$, as a function of the number of non-zero components $s$ in $x$. Success is deemed to occur if the relative $L_2$ error between the estimated $x$ and true $x$ is less than $10^{-4}$. Right: antennas placed uniformly, with large inter-element spacing, Left: antennas placed randomly over 4 m aperture.
Reconstruction, noisy

Fig. 5. Results for noisy data $y = \Phi x + z$, based on 100,000 runs, considering 30 dB SNR. As a function of the number of sources $s$, we compute the probability that Lasso [24] yields a fraction error $\|\Phi (x^* - x)\|_2 / \|\Phi x\|_2 \leq 0.05$ (5% error). Right: antennas placed uniformly, with $\lambda/2$ inter-element spacing. Left: antennas placed randomly over 4 meter aperture.
Scattering media

Fig. 6. Geometry of 19 perfectly conducting spheres placed in the presence of a 21-element 1 meter array, with $\lambda/2$ inter-element spacing.
Fig. 7. Probability of successfully recovering $\hat{x}$ as a function of $s$, based on 100,000 randomly constituted realizations. Two $m = 21$ element array designs are considered: (i) placed in the presence of 19 spherical scatterers (Figure 6), with uniform $\lambda/2$ inter-element spacing (1 meter physical aperture); (ii) the arrays placed in vacuum, spaced at random over a 4 meter aperture. Success is deemed to occur if the relative $\ell_2$ error between the estimated $\hat{x}_*$ and true $x$ is less than $10^{-4}$. 
4. **Signal estimation algorithms and performance bounds**
   - Existing CS theory enables quantitative assessment of the performance of the sensor array in an arbitrary environment in terms of the number of sources and sensors and the noise level.
Summary

1. Random sensor array is a specific case of natural CS realization
2. Applying CS theory enables us to make explicit statements about the performance of random arrays as a function of N, M, s and noise level
3. CS reconstruction algorithms exploit data sparsity; hence, motivates new regularized algorithmic approaches in array processing
4. CS insights on side-lobe level and coherence of the projections motivates new projection designs exploiting the surrounding media
Future directions

1. Design of sensor array which exploits the properties of the propagation media
   - Calibration to infer $\Phi$
   - Noise mitigation
   - Metamaterial-based antenna lenses

2. Incorporation of new algorithms that are generally superior to the early approaches

3. Development of adaptive projections
   - Select a subset of projections

4. New sampling strategies in SAR and SAS
   - Random as well as sampling more coarsely than $\lambda/2$

5. CS on analogue signal
THANK YOU!