Super-resolution Radar

Reinhard Heckel

IBM Research (before: ETH Zurich)

March 25, 2015

Joint work with:
V. Morgenshtern (Stanford), and M. Soltanolkotabi (Berkeley)
Motivation: Radar

Goal: determine location and velocities of objects
**Time-varying** linear system $H$:

\[ y(t) = \int \int s_H(\tau, \nu) x(t - \tau) e^{i2\pi \nu t} d\nu d\tau \]

**Goal:** Determine $s_H(\tau, \nu)$ from response $y(t)$ to known probing signal $x(t)$
Model

- Spreading function consists of $S$ point scatterers, that correspond to moving targets:

$$s_H(\tau, \nu) = \sum_{j=1}^{S} b_j \delta(\tau - \tau_j) \delta(\nu - \nu_j)$$
Model

- Spreading function consists of $S$ point scatterers, that correspond to moving targets:

$$s_H(\tau, \nu) = \sum_{j=1}^{S} b_j \delta(\tau - \tau_j) \delta(\nu - \nu_j)$$

- I/O-relation for point scatterers:

$$y(t) = \sum_{j=1}^{S} b_j x(t - \tau_j) e^{i2\pi \nu_j t}$$
Spreading function consists of $S$ point scatterers, that correspond to moving targets:

$$s_H(\tau, \nu) = \sum_{j=1}^{S} b_j \delta(\tau - \tau_j) \delta(\nu - \nu_j)$$

I/O-relation for point scatters:

$$y(t) = \sum_{j=1}^{S} b_j x(t - \tau_j) e^{i2\pi \nu_j t}$$

Determine the triplets $(b_j, \tau_j, \nu_j)$ from I/O-measurement
Band and time-limitation

**In practice:** $x(t)$ bandlimited to $[0, B)$ and $y(t)$ timelimited to $[0, T)$
**Band and time-limitation**

**In practice:** $x(t)$ bandlimited to $[0, B)$ and $y(t)$ timelimited to $[0, T)$

$$s_H(\tau, \nu)$$
Band and time-limitation

In practice: \( x(t) \) bandlimited to \([0, B)\) and \( y(t) \) timelimited to \([0, T)\)

\[
\begin{align*}
s_H(\tau, \nu) & \quad \text{sinc}(\tau B)\text{sinc}(\nu T) \ast s_H(\tau, \nu)
\end{align*}
\]
In practice: $x(t)$ bandlimited to $[0, B)$ and $y(t)$ timelimited to $[0, T)$

$$s_H(\tau, \nu)$$

$$\text{sinc}(\tau B)\text{sinc}(\nu T) \ast s_H(\tau, \nu)$$

band and time-limitation

\[ \frac{1}{T} \frac{1}{B} \]
**Band and time-limitation**

**In practice:** $x(t)$ bandlimited to $[0, B)$ and $y(t)$ timelimited to $[0, T)$

$$s_H(\tau, \nu) \quad \text{sinc}(\tau B)\text{sinc}(\nu T) \ast s_H(\tau, \nu)$$
In practice: $x(t)$ bandlimited to $[0, B)$ and $y(t)$ timelimited to $[0, T)$

$$s_H(\tau, \nu) \quad \text{sinc}(\tau B)\text{sinc}(\nu T) \ast s_H(\tau, \nu)$$

Resolution achieved by classic Radar via matched filtering is $\left(\frac{1}{B}, \frac{1}{T}\right)!$
Suppose the \((\tau_j, \nu_j)\) lie on a \((1, 1)\)-grid. Recovery is a sparse signal recovery problem:

\[
\mathbf{y} = \mathbf{x} p - \text{te} i 2\pi f_p \mathbf{B}^T \mathbf{b} \mathbf{B}^T (\mathbf{B}^T) \mathbf{2}
\]

Recovery via \(\ell_1\) minimization provably succeeds provided that

\[
S \leq c (\mathbf{B}^T) / \log 4 (\mathbf{B}^T)
\]

\[\text{[Krahmer et al. 2014]}\]
Suppose the \((\tau_j, \nu_j)\) lie on a \((\frac{1}{B}, \frac{1}{T})\)-grid.
Suppose the \((\tau_j, \nu_j)\) lie on a \((\frac{1}{B}, \frac{1}{T})\)-grid

Recovery is a sparse signal recovery problem:

Recovery via \(\ell_1\) minimization provably succeeds provided that
\[ S \leq c(BT)/\log^4(BT) \] [Krahmer et al. 2014]
The super-resolution Radar problem

Sampling leads to the following I/O relation:

\[ y_p = \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}} \text{IDFT}(\text{DFT}(\{x_{\ell-p}\})e^{i2\pi k \frac{\tau_j}{T}}), \quad p = 0, \ldots, BT - 1 \]
The super-resolution Radar problem

Sampling leads to the following I/O relation:

\[ y_p = \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}} \text{IDFT(DFT}\left(\{x_{\ell-p}\}\right)e^{i2\pi k \frac{\tau_j}{T}}\text{),} \quad p = 0, \ldots, BT - 1 \]

\text{time-shift of } x_{\ell-p} \text{ by } \tau_j B
The super-resolution Radar problem

Sampling leads to the following I/O relation:

\[ y_p = \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}} \text{IDFT(DFT}\{x_{\ell-p}\}\text{e}^{i2\pi k \frac{\tau_j}{T}}), \quad p = 0, \ldots, BT - 1 \]

**Super resolution Radar problem:** Determine the continuous time-frequency shifts \( \tau_j, \nu_j \) from the samples \( y_p \)
The super-resolution Radar problem

Sampling leads to the following I/O relation:

\[ y_p = \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}} \text{IDFT}(\text{DFT} \{x_{\ell-p}\} e^{i2\pi k \frac{\tau_j}{T}}), \quad p = 0, ..., BT - 1 \]

**Super resolution Radar problem:** Determine the continuous time-frequency shifts \( \tau_j, \nu_j \) from the samples \( y_p \)

**This talk:** Provably recovery of the \( (\tau_j, \nu_j) \) via convex optimization
Special case: Frequency shifts only

If $\tau_j = 0$, the problem reduces to a line spectral estimation problem:

$$y_p = x_p \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}}, \quad p = 0, \ldots, BT - 1$$
Special case: Frequency shifts only

If $\tau_j = 0$, the problem reduces to a line spectral estimation problem:

$$ y_p = x_p \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}}, \quad p = 0, \ldots, BT - 1 $$

Recovery approaches:

- **Classical**: Proney’s method
Special case: Frequency shifts only

If $\tau_j = 0$, the problem reduces to a line spectral estimation problem:

$$y_p = x_p \sum_{j=1}^{S} b_j e^{i2\pi p \frac{\nu_j}{B}}, \quad p = 0, \ldots, BT - 1$$

Recovery approaches:

- **Classical**: Proney’s method
- **Modern**: Recovery via convex optimization [Candès, Fernandez-Granda, 2014]: Recovery is possible provided the minimum separation condition holds:

$$\left| \frac{\nu_j}{B} - \frac{\nu_i}{B} \right| \geq \frac{4}{BT}, \quad \text{for all } j \neq i$$
Do we need minimum separation?

- **Compressed sensing**: RIP guarantees that the energy of all sparse signals is preserved.
Do we need minimum separation?

- **Compressed sensing:** RIP guarantees that the energy of all sparse signals is preserved.

- **Super-resolution:** Suppose the $\nu_j$ are at equidistant positions, and $S$ is large.

\[
y_p = e^{i2\pi p \frac{\nu_j}{B}} b_j
\]

Energy is only preserved if the distance between the $\nu_j$ is sufficiently large!
Recovery results for the super-resolution Radar problem
Main result

- Random probing signal: $x_\ell$ i.i.d. $\mathcal{N}(0, 1)$
- Random signs: sign of $b_n$ is i.i.d. uniform on the complex unit sphere

Theorem

The $(\tau_j, \nu_j, b_j), j = 1, ..., S,$ can be recovered by solving a semidefinite program with probability $\geq 1 - \delta$ if

$$|\tau_j - \tau_i| \geq \frac{5}{B} \text{ or } |\nu_j - \nu_i| \geq \frac{5}{T}, \text{ for all } i \neq j$$

and if

$$S \leq cBT \log^{-3} \left( \frac{(BT)^6}{\delta} \right)$$
Main result

- Random probing signal: $x_\ell$ i.i.d. $\mathcal{N}(0, 1)$
- Random signs: sign of $b_n$ is i.i.d. uniform on the complex unit sphere

Theorem

The $(\tau_j, \nu_j, b_j), j = 1, \ldots, S$, can be recovered by solving a semidefinite program with probability $\geq 1 - \delta$ if

$$|\tau_j - \tau_i| \geq \frac{5}{B} \text{ or } |\nu_j - \nu_i| \geq \frac{5}{T}, \text{ for all } i \neq j$$

and if

$$S \leq cBT \log^{-3} \left( \frac{(BT)^6}{\delta} \right)$$

- Essentially optimal
Recovery approach: “A continuous counterpart of $\ell_1$-minimization”
Recovery approach: “A continuous counterpart of $\ell_1$-minimization”

Proof based on analyzing the dual: Construction of a dual certificate (dual polynomial)
Comments

- Recovery approach: “A continuous counterpart of $\ell_1$-minimization”
- Proof based on analyzing the dual: Construction of a dual certificate (dual polynomial)
- Stable
Recovery approach: “A continuous counterpart of $\ell_1$-minimization”

Proof based on analyzing the dual: Construction of a dual certificate (dual polynomial)

Stable

Super-resolution Radar on a grid

- Suppose the $(\tau_j, \nu_j)$ lie on a fine grid:

Our results guarantee success of $\ell_1$-minimization
Future work

Implementation in hardware:
Conclusion

**Problem:** Estimation of the time-frequency components of a signal that is $S$-sparse in the *continuous* dictionary of time frequency shifts of a random function.
Problem: Estimation of the time-frequency components of a signal that is $S$-sparse in the *continuous* dictionary of time frequency shifts of a random function

Main result: Recovery via convex optimization provably succeeds provided that:

- Minimum separation condition holds
- Number of measurement linear (up to log-factor) in $S$
Conclusion

**Problem:** Estimation of the time-frequency components of a signal that is $S$-sparse in the *continuous* dictionary of time frequency shifts of a random function

**Main result:** Recovery via convex optimization provably succeeds provided that:
- Minimum separation condition holds
- Number of measurement linear (up to log-factor) in $S$

For more details:
Conclusion

Problem: Estimation of the time-frequency components of a signal that is $S$-sparse in the continuous dictionary of time frequency shifts of a random function

Main result: Recovery via convex optimization provably succeeds provided that:

- Minimum separation condition holds
- Number of measurement linear (up to log-factor) in $S$

For more details:

Thank you!